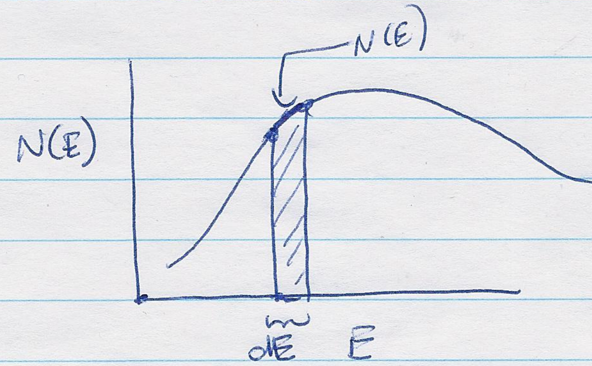
# The Dist'n fcn of an Ideal Gas

assume $f=3$, ie a monatomic gas so that only translational motions are relevant.

want: $N(E)$




a small element is

$$dN(E) = N(E)\,dE$$

The total # of particles would be

$$N = \int dN(E) = \int N(E)\,dE$$

$dN(E)$ = # of particles with energies between $E$ + $E+dE$

For a discrete case, we had $\dfrac{N_i}{N} = \dfrac{g_i}{Z} e^{-E_i/kT}$ where $i$ refers to the energy level and $g_i$ is the degeneracy of that level. (statistical wt)

For a continuous distribution, then

$$\frac{N(E)}{N} = \frac{g_E}{Z} e^{-E/kT}$$

(the statistical wt can change with E

so $dN(E) = \dfrac{dg_E}{Z} e^{-E/kT}$ (1) For a fixed T, there is a range of E

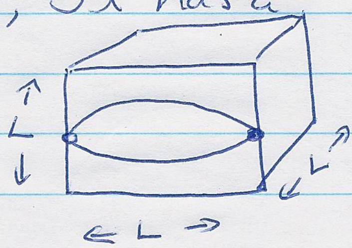
the next goal is to find expressions for $dg_E$ and $Z$ in terms of E

Let's start with $dg_E$

$\boxed{dg_E}$ We want to know how many states are available for a given E (ie between E + E+dE)

Consider a particle inside a box, It has a de Broglie wavelength, $\lambda$, ∋

$$p = \frac{h}{\lambda}$$



what is the smallest box size, L, that can contain a particle? $L = \frac{n\lambda}{2}$

(Recall our early development of Kinetic Theory. We considered a box size ∋ a particle hit a wall and made it back to the other side again.)

In 1-Dimension:

$$p = \frac{nh}{2L}$$

n an integer, ie a "quantum #" for free space

In 3-D:

SEE FIGURE

$$p^2 = p_x^2 + p_y^2 + p_z^2 = (n_x^2 + n_y^2 + n_z^2)\left(\frac{h}{2L}\right)^2$$

For translational motion only:

$$E = \frac{1}{2} m v^2 = \frac{1}{2}\frac{p^2}{m} = \frac{1}{2}\frac{(n_x^2 + n_y^2 + n_z^2)}{m}\left(\frac{h}{2L}\right)^2 = \frac{r^2 h^2}{8m}\frac{1}{L^2} \qquad (2)$$

where $r^2 = n_x^2 + n_y^2 + n_z^2$

$\therefore\ r = \frac{L}{h}(8mE)^{1/2}$ and $dr = \frac{L}{h}(8m)^{1/2}\frac{1}{2}E^{-1/2}\,dE$

what is the volume in "state space"?

$$\boxed{dg_E} = \frac{1}{8}(4\pi r^2)\,dr = \frac{1}{8}(4\pi)\left[\frac{L^2}{h^2}(8mE)\right]\left[\frac{L}{h}(8m)^{1/2}\frac{1}{2}E^{-1/2}\,dE\right]$$

$n_x, n_y, n_z \geq 0$ so looking at one octant only

$$\boxed{= \frac{2\pi V}{h^3}(2m)^{3/2}E^{1/2}\,dE}, \qquad (3) \qquad V = L^3$$

Now let's consider the partition fcn $\boxed{Z}$
I will write $Z_{tr}$ to indicate that I am considering the partition fcn for translational states only. For discrete energy states, $i$,

$$Z_{tr} = \sum_i e^{-E_i/kT}$$ but from Eq'n (2), $E = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$

so
$$Z_{tr} = \left[\int e^{-\frac{h^2 n_x^2}{8mkTL^2}} dn_x\right]\left[\int e^{-\frac{h^2 n_y^2}{8mkTL^2}} dn_y\right]\left[\int e^{-\frac{h^2 n_z^2}{8mkTL^2}} dn_z\right]$$

Now carry out the integration, noting that $\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

to find $Z_{tr} = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}$ (4) let $v_q = \left(\frac{h^2}{2\pi mkT}\right)^{3/2}$ be the "quantum volume"

then $Z_{tr} = V/v_q$ see also eq'n 6.82, 6.83 text
For a given $T$, $v_q$ is the smallest volume that can contain a particle

We now have both $dg_E$ (eqn 3) and $Z$ (eqn 4), so can return to Eqn (1)

$$dN(E) = \frac{N\,dg_E}{Z} e^{-E/kT} = N \frac{2}{\pi^{1/2}(kT)^{3/2}} E^{1/2} e^{-E/kT} dE$$ Dist'n fcn in KE

# particles between $E$ and $E + dE$

noting that $E = \frac{1}{2}mv^2$ we can change to $v$

$$\boxed{dN(v) = N\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-\frac{1}{2}mv^2/kT} dv}$$ Maxwell-Boltzmann vel. dist'n

# of particles between $v$ and $v + dv$

where $\int dN(v) = N$

cf. eq'n 6.50, Fig 6.13 text