PHYS 815 - Physics of Degenerate Matter

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1 Introduction

Modern astrophysics is a window into incredible cosmic events, pushing into densities, speeds, temperatures, and field strengths far beyond the human "normal". These extreme conditions provide a proving ground for theories of dense matter physics, nuclear physics, and particle physics. All of these are utilized in the study of degenerate matter physics.

Degenerate matter is a state of extremely densely packed fermions which support themselves through particle interactions and fermion kinetic energy. To uphold this density, these fermions must have enough energy to satisfy the Pauli exclusion principle. Degenerate matter is found in stars such as white dwarfs and neutron stars which are at the end stages of life and are no longer undergoing thermonuclear burning, meaning these stars must support themselves with the degeneracy pressure of their individual atoms and nuclei pressing against each other (An Introduction to Modern Astrophysics, 2006). We will explore the deep physics involved with this state of matter with a major focus on degenerate stars, and in particular, neutron stars. As we travel into higher densities in these systems, we will refer to densities ranging from typical earthbound Fe^{56}_{26} at $\rho = 7gcm^{-3}$ to densities greater than $\rho \approx 10^{16}gcm^{-3}$. This, amazingly, will cover the range of densities found on the surface of white dwarfs, to the very center of neutron stars.

Before a more complete understanding of degenerate matter physics can be achieved, it is important to understand the equations of state in a zero temperature Fermi gas. Using the material derived in Sec. 2, we first examine densities below which "neutron drip" occurs, which is roughly below $\rho_{ND} \approx 4 \times 10^{11} g\ cm^{-3}$ (The Physics of Compact Objects, 1983). For reference, neutron drip is where relativistic electrons penetrate some nucleon that already has all energy levels filled, and as such the collision produces a neutron that "drips" out of the nucleon, becoming a free neutron (Acevedo et al, 2019). At densities beyond this point, referred to as the "neutron drip line" the physics becomes complicated by particle interactions and relativistic effects. Generally, the values calculated above neutron drip are appropriate for white dwarf stars and the atmosphere and outer crust of neutron stars (see Fig. 2), much as was found in class (The Physics of Compact Objects, 1983). We will not focus much on this material beyond adding some simple corrections, as it was covered in depth during class.

2 Fermi Gases - A Review

Fermi gases are the very base with which we can begin to understand dense matter physics. Nearly everything onward will build off the basic concepts developed here. Indeed, it will be shown that white dwarf structure can be understood with just these principles show here (plus some corrections). We first assume we have some completely degenerate ideal gas, at zero temperature, otherwise we must account for electrostatic interactions between these particles. The distribution function of these particles can be approximated in this case with a Maxwell-Boltzmann distribution, which for fermions is as follows

$$f(E) = \frac{1}{\exp\left(\frac{\mu - E}{kT}\right) + 1} \tag{1}$$

However, it is clear that when $T = 0$, $\mu = E_F$ (Terzic, 2007),

$$f(E) = \begin{cases} 1 & \text{if } E_F \geq E \\ 0 & \text{if } E_F \leq E \end{cases} \tag{2}$$

Where the Fermi energy, E_F is the difference between the highest and lowest single particle states of these fermions, which we assume are non-interacting. Being fermions, these particles obey Fermi-Dirac statistics, and are typically idealized as a gas of electrons, neutrons, protons, or some other half-integer spin particle, making them particularly useful during a discussion of dense matter physics. For the purposes of this review, we will discuss the three dimensional and (for now) non-relativistic case, and assume the gas is of electrons at zero temperature and no electrostatic interaction. We know the density of states in a 3D volume of free electrons is

$$g(E) = \frac{1}{2\pi^2}(\frac{2m_e}{\hbar^2})^{3/2}E^{1/2} \tag{3}$$

And, knowing the number density of any particle is given by

$$n \equiv \int_0^\infty g(E)f(E)dE \tag{4}$$

we can find n_e easily as follows

$$n_e = \frac{1}{2\pi^2}(\frac{2m_e}{\hbar^2})^{3/2}\int_o^{E_F} E^{1/2} = \frac{1}{3\pi^2}(\frac{2m_e}{\hbar^2})^{3/2}(E_F)^{3/2} \tag{5}$$

and therefore, the Fermi energy, E_F is found to be

$$E_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{3/2} \tag{6}$$

In a degenerate system, nearly all the energy levels will be filled and the maximum energy of a filled level will be E_F, the Fermi energy. We can relate this to the Fermi temperature $T_F = E_F/k$, where if $T \ll T_F$ then thermal effects will be small (The Physics Of Compact Objects, 1983). Therefore, our assumption of $T = 0K$ is not a bad approximation, as we will nearly always have $T \ll T_F$ in this report.

We can also relate the Fermi energy to the Fermi momentum via the following relation (Terzic, 2007):

$$E_F = \frac{p_F^2}{2m_e} \quad \rightarrow \quad p_F = \sqrt{E_F 2m_e} \tag{7}$$

And, similarly, we can describe the Fermi velocity to be,

$$v_F = \frac{p_F}{m_e c} \tag{8}$$

The Fermi velocity is particularly useful for the coming analysis, as we will use it as a "relativity parameter" to allow us to vary the equations of state we will be deriving with how relativistic the gases are (The Physics of Compact Objects, 1983). Indeed, to build the equations of state for degenerate Fermi gases of electrons in the next section, we will ultimately integrate over this Fermi velocity to find the degeneracy pressure.

3 Densities below "Neutron Drip"

White dwarfs are stars at the end of their lives, are typically a few thousand kilometers in diameter, and generally have a mass similar to the sun. This means that they have average densities on the order of $10^6 gcm^{-3}$. Most importantly, they no longer produce enough thermal energy to hold themselves up against gravitational collapse. Thus, they must rely on the degeneracy pressure of the fermions that make up their constituent matter to support themselves. We seek to understand the equation of state (EOS) for this system, and being below the density of neutron drip (to be discussed later), this means we will predominately be seeking to understand electron degeneracy pressure. We can understand this as a combination of the effects of both the Pauli exclusion principle and the Heisenberg uncertainty principle, $\Delta x \Delta p \approx \hbar$. From the Heisenberg uncertainty principle it is clear that if we have well confined electrons in a small volume of space, we must have an extremely high uncertainty in the momentum of these electrons. If we assume we have a Fermi gas of fully degenerate electrons, as we do in a white dwarf, with some number density n_e, then from arguments of geometry, we expect the spacing between particles to be roughly $n_e^{-1/3} \approx \Delta x$ (An Introduction to Modern Astrophysics, 2006). Therefore,

$$p_x \approx \Delta p_x \approx \frac{\hbar}{\Delta x} \approx \hbar n_e^{1/3} \tag{9}$$

and so, because we care about a 3D gas, and $p^2 = p_x^2 + p_y^2 + p_z^2 \quad \rightarrow p^2 = 3p_x^2$, we have

$$\begin{aligned} p &= \sqrt{3} p_x \\ &= \sqrt{3} \hbar n_e^{1/3} \\ &= \sqrt{3} \hbar \left[(Z/A) \frac{\rho}{m_H} \right]^{1/3} \end{aligned}$$

Clearly, we can see that in these dense matter systems, the momentum, average particle spacing, and density will be deeply intertwined very generally (An Introduction to Modern Astrophysics, 2006). This is important to keep in mind for the rest of this report.

3.1 Complete Degeneracy of Ideal Fermi Gas

We again consider the Fermi gas of electrons, as discussed in Sec. 2. Again, as shown above, taking $T = 0$ is a fairly good approximation for densities below neutron drip. One major difference is that we now allow some particles to be relativistic. As such, we can define our Fermi energy and Fermi momentum to be

$$E_F \equiv (p_f^2 c^2 + m_e^2 c^4)^{1/2} \tag{10}$$

And, using Eqn. 4 and Eqn. 7, we can find the number density of electrons in this system (Baym, Bethe, & Pethick, 1971),

$$n_e = \frac{2}{\hbar^3} \int_0^{p_F} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3 \tag{11}$$

This can be simplified by using the Fermi velocity found in Sec. 2, and we have $n_e = (3\pi^2 \lambda_e^3)^{-1} v_F^3$, where λ_e is simply the electron Compton wavelength,

$$\lambda_e = \frac{\hbar}{m_e c} \tag{12}$$

and $v_{F,e}$ is being used as a relativity parameter for the electron gas. Next, we must find the pressure in the system due to these degenerate electrons (Baym, Bethe, & Pethick, 1971). We know that the pressure from some number density of particles in a volume for some momentum p is

$$P = \frac{1}{3} \int_0^{\infty} npv \ dp \tag{13}$$

Where the $\frac{1}{3}$ is from geometrical effects, and we therefore find that for our cold Fermi gas of degenerate electrons

$$\begin{aligned} P_e &= \frac{1}{3} \frac{2}{\hbar^2} \int_0^{p_F} \frac{4\pi p^4 c^2}{(p^2c^2 + m_e^2 c^4)^{1/2}} dp \\ &= \frac{8\pi m_e^4 c^5}{3h^3} \int_0^{v_{F,e}} \frac{v_{F,e}^4 dv_{F,e}}{(1 + v_{F,e}^2)^{1/2}} \\ &= \frac{m_e c^2}{\lambda_e^3} \phi(v_{F,e}) \end{aligned}$$

Where the function $\phi(v_{F,e})$ is the solution to the above integral, which depends on the Fermi velocity (Baym, Bethe, & Pethick, 1971). We care about the limits of $v_F \ll 1$ and $v_F \gg 1$, which represent non-relativistic electrons and relativistic electrons respectively (Baym, Bethe, & Pethick, 1971). These limits give us

$$\phi(v_{F,e}) \rightarrow \frac{v_F^5}{15\pi^2} \quad \text{if } v_F \ll 1 \tag{14}$$

$$\phi(v_{F,e}) \rightarrow \frac{v_F^4}{12\pi^2} \quad \text{if } v_F \gg 1 \tag{15}$$

Clearly, we see here that the degeneracy pressure is dependent on the Fermi velocity, and therefore the Fermi momentum. At the beginning of this report, we saw a clear link between the momentum, and the corresponding density. Therefore, we can say that there is a direct link between the resulting degeneracy pressure and the density at some depth, and so we can write the equation of state in the polytropic form of

$$P = K\rho^\gamma \tag{16}$$

For a degenerate Fermi gas of electrons, we know $\rho \approx n_e m_u \mu_e \approx 10^6 \mu_e v_{F,e}^3 \; gcm^{-3}$, where m_u is the atomic mass unit and where μ_e is the mean molecular weight per electron (The Physics of Compact Objects, 1983). Additionally, it is now clear that the relativistic and non-relativistic limits depend on the density. These densities are $\rho \gg 10^6 gcm^{-3}$ and $\rho \ll 10^6 gcm^{-3}$ respectively (The Physics of Compact Objects, 1983). As before, must consider the limiting relativistic and non-relativistic cases when finding the values of K and γ. These are as follows

$$\textbf{non-relativistic: } \gamma = \frac{5}{3}, \quad K = \frac{1.0036 \times 10^{13}}{\mu_e^{5/3}} \quad \text{if } \rho \ll 10^6 gcm^{-3} \tag{17}$$

$$\textbf{relativistic: } \gamma = \frac{4}{3}, \quad K = \frac{1.2435 \times 10^{15}}{\mu_e^{4/3}} \quad \text{if } \rho \gg 10^6 gcm^{-3} \tag{18}$$

for a degenerate Fermi gas made of non-relativistic electrons and relativistic electrons respectively. This ideal Fermi gas equation was used by Chandrasekhar when he sought to understand the structure of white dwarfs in equilibrium (Chandrasekhar, 1931). There are some corrections due to electrostatic interactions which we will not consider here. This is ultimately the end of our investigation into white dwarf stars, as beyond a few corrections, the analysis done up to this point can provide a relatively accurate equation of state, and provides us with a good understanding of the internal structure of the degenerate star. In some cases, Sec. 3.2, and Sec. 3.3 can apply in the core of white dwarf stars.

Similarly to the above analysis, we can find an equation of state for a degenerate Fermi gas of neutrons, which will become exceptionally useful as we push to higher densities, such as those seen in Fig 2. Like the degenerate electron gas, we find the density to be $\rho = m_n n_n \approx 10^{15} v_{F,n}^3 \; gcm^{-3}$ (The Physics of Compact Objects, 1983). The equation of state follows the same polytropic form as the degenerate electron gas, as found in Eqn. 16. The values of K and γ again change depending on whether the gas is relativistic or not, as seen below

$$\textbf{non-relativistic: } \gamma = \frac{5}{3}, \quad K = 5.3802 \times 10^9 \quad \text{if } \rho \ll 10^{15} gcm^{-3} \tag{19}$$

$$\textbf{relativistic: } \gamma = \frac{4}{3}, \quad K = 1.2293 \times 10^{15} \quad \text{if } \rho \gg 10^{15} gcm^{-3} \tag{20}$$

We must be wary of this equation of state for degenerate neutrons however, because major corrections need to be added as we increase the density.

3.2 Ideal Inverse Beta Decay

Everything we have discussed thus far works well for densities below roughly $\rho \approx 10^7$, which is the point at which the degenerate Fermi sea of electrons becomes fully relativistic (The Physics of Compact Objects, 1983). That is to say, as we enter the outer crust of neutron stars, these relativistic electrons add important

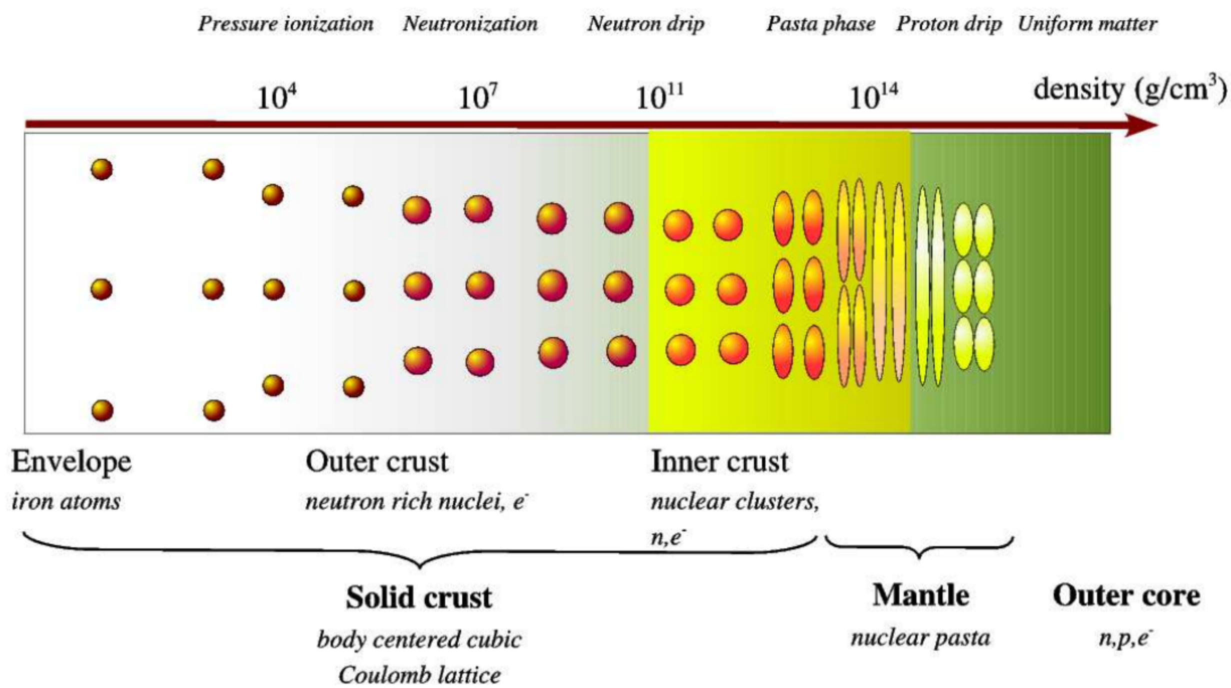

Figure 1: A rendition of a neutron star interior, not to scale, showing the densities at which various forms of nuclear matter appear. The density scale roughly follows the analysis we will use in the rest of the report. From (Chamel, 2016).

corrections to our equation of state. If an electron has enough energy to balance the difference in mass between the proton and the neutron ($(m_n - m_p)c^2 = 1.29MeV$), the process of Inverse β-decay may take place. If this occurs, this process will produce an excess of free neutrons by changing protons into neutrons. Ultimately, this process can be negated by normal β-decay, but if β-decay cannot occur, this "sea" of free neutrons will be created, and nuclei will be added in larger and larger numbers to be existing nuclei turning the lattice of iron nucleons in the atmosphere into a dense lattice is neutron rich nuclei as we proceed further into the neutron star. It turns out that β-decay can be blocked at high enough densities (The Physics of Compact Objects, 1983). Inverse β-decay produces the following reaction:

$$p + e^- \rightarrow n + \nu_e \tag{21}$$

Where we assume that the neutrino escapes due to its small mass. We can now "redo" the above analysis, for a gas of free electrons, neutrons, and protons. Generally, the neutrons and protons are bound, and we will consider this case in Sec. 3.3. However, if we assume that the protons, neutrons, and electrons are in equilibrium, and are in a free gas, then

$$\mu_e + \mu_p = \mu_n \tag{22}$$

where μ if the chemical potential of a particle. We then incorporate the "relativity parameter" from Eqn. 8, giving us a different value for each particle in consideration, due to their different masses. This also allows us to vary the speeds with which they are travelling. Now, clearly, we can use the fact that $E_{F,e} = \mu_e = (p_{F,e}^2 c^2 + m_e^2 c^4)^{1/2}$, and we can relate the masses of the particles in equilibrium as follows

$$m_e(1 + v_{F,e}^2)^{1/2} + m_p(1 + v_{F,p}^2)^{1/2} = m_n(1 + v_{F,n}^2)^{1/2} \tag{23}$$

Additionally, because of charge neutrality, we can see that $n_e = n_p$ and so,

$$\frac{v_{F,e}^3}{3\pi^2 \lambda_e^3} = \frac{v_{F,p}^3}{3\pi^2 \lambda_p^3} \tag{24}$$

And like before, we easily find the degeneracy pressure and total number density to be (The Physics of Compact Objects, 1983),

$$P = \frac{m_e c^2}{\lambda_e^3}\phi(v_{F,e}) + \frac{m_p c^2}{\lambda_p^3}\phi(v_{F,p}) + \frac{m_n c^2}{\lambda_n^3}\phi(v_{F,n}) \tag{25}$$

$$n = \frac{v_{F,p}^3}{3\pi^2\lambda_p^3} + \frac{v_{F,n}^3}{3\pi^2\lambda_n^3} \tag{26}$$

We can now use this as an opportunity to determine the density at which neutron drip begins to occur. Indeed, we can use the equations we have just calculated to see how the number of neutrons change as we increase the density. This will become particularly important, as our focus in the later sections quickly shifts towards neutron star interiors. First, we recognize that the minimum density at which free neutrons may be found in as a free gas outside of our nuclei will occur when we set $v_{F,n} = 0$ in Eqn. 23. Using the fact that at this density, protons will be relativistic (The Physics of Compact Objects, 1983), we can see

$$m_e(1 + v_{F,e}^2)^{1/2} = m_n - m_p \tag{27}$$

Solving for $v_{F,e}$ and using the number density of this free gas found in Eqn. 24, and using Eqn. 26, we get

$$n = \frac{1}{3\pi^2\lambda_e^3}\left[(\frac{m_n - m_p}{m_e})^2 - 1\right]^{3/2} \tag{28}$$

and so we arrive at which the first free neutrons may be found, using

$$\rho \approx n m_p \approx 1.2 \times 10^7 gcm^{-3} \tag{29}$$

So, above this density, we will have an increasing fraction of free neutrons, significantly changing the physics in the following sections. Additionally, we can ask where the nuclear drip line actually occurs. Using a similar method as above, we find that

$$n_n = \frac{2^{3/2}}{3\pi^2\lambda_n^3}\left[\frac{(m_n - m_p)^2 - m_e^2}{m_n^2}\right]^{3/4} \tag{30}$$

and so, we find that the nuclear drip line begins around

$$\rho_{ND} \approx m_n n_n \approx 4.3 \times 10^{11} gcm^{-3} \tag{31}$$

3.3 Baym - Pethick - Sutherland (BPS) Equation of State

In reality, we don't find an ideal Fermi gas of free electrons, protons, and neutrons, but rather find that the protons and neutrons remain bound at densities from $10^7 \leq \rho \geq 4 \times 10^{11} gcm^{-3}$, where neutron drip begins. This covers regions in the neutron star from the outer crust to the inner crust. We also find the nuclei (entirely composed of Fe_{26}^{56} at these densities) in a Coulomb lattice, minimizing the electrostatic energy of the system. As densities increase, this lattice (typically considered to be of the form *bcc*, or body-centered cubic) is permeated by free neutrons and relativistic electrons, and thus the lattice energy changes, again changing the equation of state (Baym, Pethick, Sutherland, 1971). We will follow closely the derivation by Baym, Pethick, and Sutherland, to come to this final equation of state.

Nuclear matter, according to BPS theory, now changes with density in the following way. At low density (say, that of a white dwarf interior or neutron star atmosphere) where $\rho \approx 10^6$, the nuclei will be made of mostly Fe_{26}^{56} and arranged in a *bcc* lattice. As previously discussed, at around $\rho \approx 10^4 gcm^{-3}$, electrons are mostly free, and by $\rho \approx 10^7 gm^{-3}$, we know these free electrons will be relativistic, and inverse β-decay will occur, liberating neutrons, and as the density continues to increase, the fraction of free neutrons within this lattice continues to increase, decreasing the effect Coulomb forces play on the equation of state. Past the nuclear drip line of $\rho_{ND} \approx 4.3 \times 10^{11} gcm^{-3}$, the continuum neutron states start to become populated, and as the density surpasses $\rho \approx 4 \times 10^{12} gcm^{-3}$, the pressure becomes more dominated by the neutrons than the electrons. Ultimately past $\rho_0 \approx 2.4 \times 10^{14} gcm^{-3}$, we get a liquid of protons, neutrons, and electrons (Baym, Pethick, Sutherland, 1971). As the density is continued to be raised, strange particles appear and supposedly coexist.

To begin to derive the equation of state that describes this, we first notice that it is the lattice Coulomb energy that begins to dominate the nuclear size (Baym, Pethick, Sutherland, 1971). In doing so, we understand that the final pressure of the system will depend on the same pressure found in Eqn. 16, for either electrons or neutrons, depending on the density as discussed above, with some small corrections, dependant on the density, which outlines how the Coulomb repulsive forces change the pressure between $10^7 gcm^{-3} \leq \rho \leq 10^{11} gcm^{-3}$. We can write the pressure as

$$P = P_e + P_L \tag{32}$$

Recall that the total energy of the system can be used to evaluate the pressure, as was done for a much simpler system in Sec. 2. Here, we simply have a new contribution to the energy from the Coulomb force, and we first write out the energy density contribution of the lattice. Assuming the lattice is a Body-Centered Cubic lattice, as previously discussed, we have

$$E_{tot} = E_p + \epsilon_L = n_n E(A, Z) + \epsilon_e + \epsilon_L \tag{33}$$

where our previously we have been assuming the energy of only free particles as E_p, and the new contribution from the lattice is ϵ_L (Baym, Pethick, Sutherland, 1971). We can easily find this contribution by using the Wigner-Seitz approximation (Nandi Debades, 2007). This approximation assumes the nucleon to be a sphere of some radius r_0, and can relate the radius to our number density with $r_0 = (3Z/4\pi n_e)^{1/3}$ (Baym, Pethick, Sutherland, 1971). The energy needed to assemble this lattice, and therefore the energy acting on individual particles in the lattice, is $\epsilon_L = E_{e,e} + E_{e,i}$, where $E_{e,e}$ accounts for electron-electron interactions, and $E_{e,i}$ accounts for electron-ion interactions. Ultimately, we find the lattice energy to be

$$\epsilon_L = -\frac{9}{10}\frac{Z^2 e^2}{r_0} \tag{34}$$

$$= -1.44 Z^{2/3} e^2 n_e^{4/3} \tag{35}$$

This allows us to find the lattice pressure component to be $P_L = \frac{1}{3}\epsilon_L$. In practice, because the values of A, and Z are discrete (causing discontinuities ρ), it is typical to minimize the Gibbs free energy of the system at a constant pressure and zero temperature, an obtain the maximum pressures and densities of specific elements through this technique (The Physics Of Compact Objects, Baym, Pethick, Sutherland, 1971).

This concludes the corrections that must be make in white dwarfs and neutron star atmospheres and outer crusts (see Fig. 2. We will now travel below the nuclear drip line, where we find physics begins to change significantly.

4 The Nuetron Drip Line

As we leave behind the outer crusts of neutron stars, and pass through the neutron drip line into the inner crust and beyond, it is imperative that we understand what the neutron drip line represents.

As before, we allow N to represent the number of neutrons in a nucleon, Z to be the number of protons in a nucleon, and A to be the mass number. As our seed atom undergoes neutron capture, the value of N will increase by one. If the combination of N and Z do not result in a stable atom, the atom will experience a β-decay, changing the neutron into a proton. Using this process, a seed atom can work its way up to more massive nucleons. However, if the β-decay process is blocked, the nucleon will continue to undergo neutron capture until it there are no longer any bound states for the neutron to fill in the atom. When this occurs, neutron capture will no produce a free neutron outside of the body of the nucleon, creating a free neutron

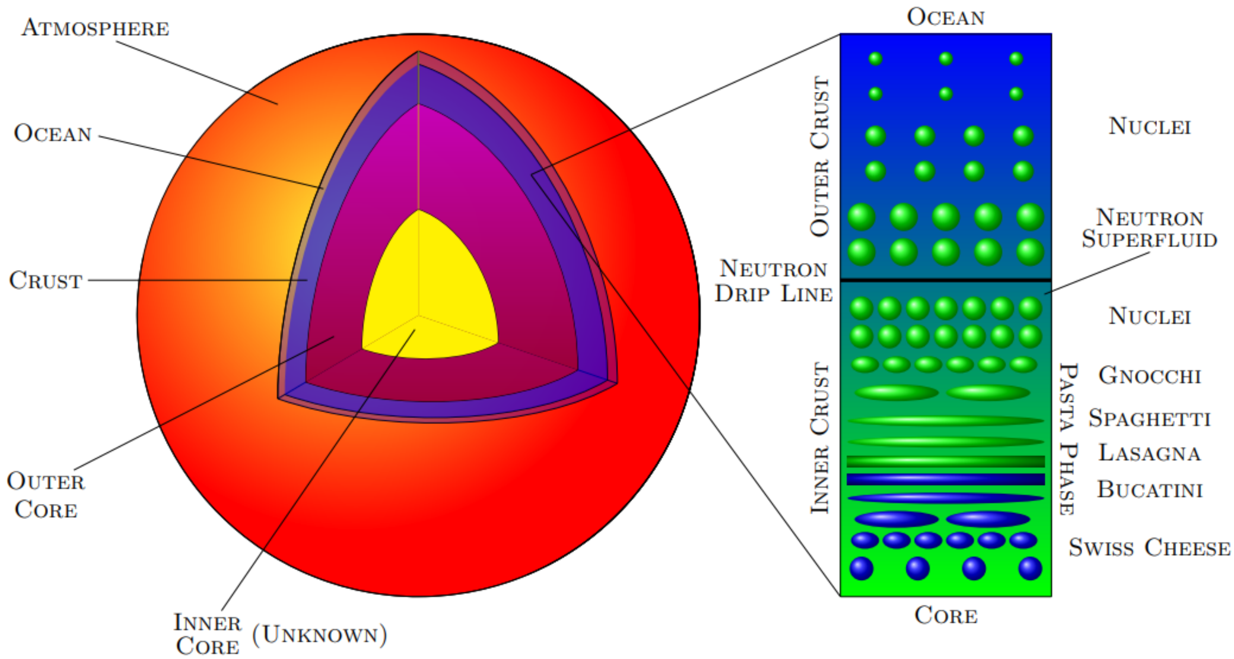

Figure 2: A rendition of a neutron star interior, not to scale. On the right, an artists depition of the stages of matter above and below the neutron drip line. From (Acevedo et al, 2019).

gas that grows in density as the pressure and density increase. In neutron stars, β-decay is blocked by a filled Fermi sea of electrons. As such, below the value of $\rho_{ND} \approx 4.3 \times 10^{11} gcm^{-3}$, a free neutron gas begins to appear (Acevedo et al, 2019). This will change the physics beyond this point, where we now see a lattice of neutron rich nuclei and a free neutron gas. In the following sections, we will deal with the nucleon-nucleon interactions that this free gas produces.

5 Densities Above "Neutron Drip"

Densities below neutron drip are characterized by the aforementioned free neutron gas. As we pass below the neutron drip line, we enter densities corresponding to the inner core of neutron stars. This analysis will briefly follow the Baym-Bethe-Pethick (BBP) method. This method is based on a "compressible liquid drop" model of an atom (Baym, Bethe, & Pethick, 1971). We will denote values in a nuceli with a subscript N, and values in the free neutron gas with a subscript G. We first write the baryon density as

$$n = An_N + (1 - V_N n_N)n_n \tag{36}$$

where n_N is the number density of nuclei, and $n_n = \frac{N_n}{V_n}$, and finally, V_N is the volume of a nucleus (Baym, Bethe, & Pethick, 1971). Then, we acknowledge that there will be some new pressure contribution from this free neutron gas in the system. We then make a number of assumptions. First, we recognize that we must find the optimal A for nuclei by finding the minimum energy with respect to A, giving us

$$\frac{\partial}{\partial A}(\frac{E}{A}) = 0 \tag{37}$$

Second, we recongize that the nuclei must not be capable of β-decay, as we are beyond the neutron drip line, so our nuclei must have a stable Z (Baym, Bethe, & Pethick, 1971). After some work, this gives us,

$$\mu_e = \mu_{n,N} - \mu_{p,N} \tag{38}$$

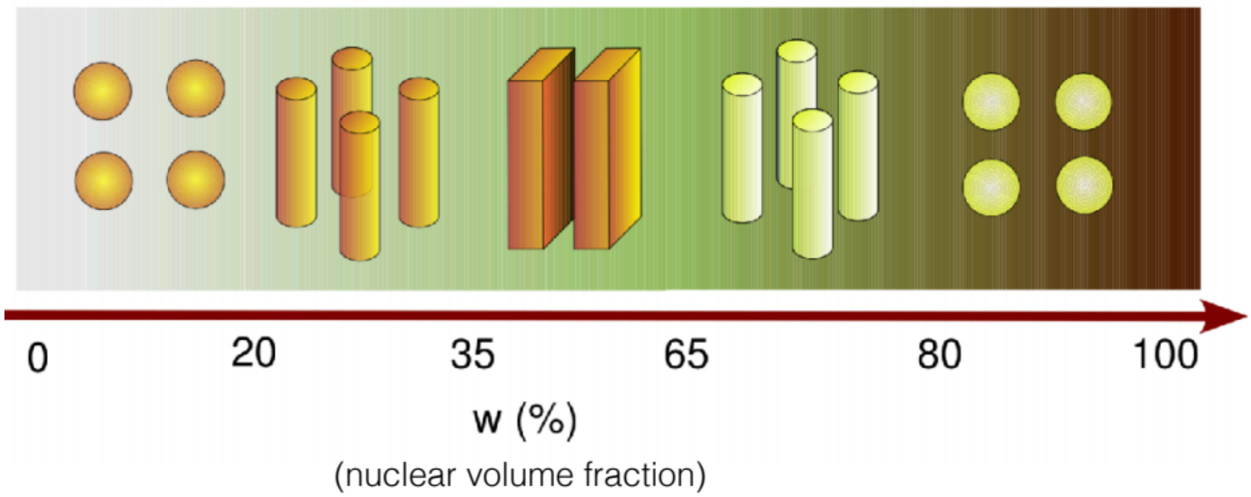

Figure 3: A depition of how matter in the inner crust changes with density. Here $W = \rho/\rho_0$. From (Chamel, 2016).

relating the chemical potential energy of the particles found in this system. We then assume that the free neutron gas must be in equilibrium with neutrons in the nuclei (Baym, Bethe, & Pethick, 1971). Therefore,

$$\mu_{n,N} = \mu_{n,G} \tag{39}$$

Finally, we assume that the pressure of the gas is in equilibrium with the pressure of the nuclei. This gives us $P_G = P_N$ (Baym, Bethe, & Pethick, 1971). Using these equations, we can find the pressure contribution from the free neutron gas in the inner core to be of the form,

$$P = P_e + P_L + P_G \tag{40}$$

where P_G is the pressure due to the free neutrons in a gas, and after utilizing the four conditions laid out above, we can arrive at a value for P_G of

$$P_G = n_n \mu_{n,G} - \epsilon_n \tag{41}$$

Where ϵ_n is the energy contribution from the free neutron gas.

Ultimately, the density of the neutron star continues to increase as we travel closer to the core. At this point, we are nearing the region know as the outer core. However, before we get there, we must pass through a region filled with "nuclear pasta" (Acevedo et al, 2019).

5.1 Nuclear Pasta

In the last few hundred meters of the inner crust, the extreme densities cause the shape of nuclear matter to become very Italian in nature. This is due to an interplay between the attractive nuclear force acting at very short ranges, and the repulsive Coulomb force, as we have already discussed (Sagert, 2016). This is also due to the fact in the compressible liquid drop model, a new energy term appears, E_S. This is the surface energy of a nuclei, and follows the relation

$$\epsilon_S = u\sigma d/r \tag{42}$$

where u is the fraction of space filled with dense matter, σ is the density dependant surface tension, d is the dimensionality, and r is the radius of this nucleus (Iida et al, 2004). When this term is added to the total energy to form

$$E_{tot} = E_p + \epsilon_L + \epsilon_S = n_n E(A, Z) + \epsilon_e + \epsilon_L + \epsilon_S \tag{43}$$

This total energy changes depending on the density (Ravenhall et al, 1983). If you were to plot the total energy against the density, up to the nuclear saturation density, you would see a function with many minimums, resulting from the fact that the surface energy in our model can help minimize the total energy by changing the value of d. This results in exotic shapes of matter, as can be seen in Fig. 3. First, as expected, the nuclei are spherically shaped, before it becomes energetically favourable to form long tube-like structures (often referred to as nuclear spaghetti). It then becomes energetically favourable to for long thin layers, similar to lasagna, to form (Ravenhall et al, 1983). Both the spaghetti and lasagna forms are surrounded by a dense neutron gas. As we continue to increase the density, this paradigm reverses, and the dense nuclear matter now surrounds long, thin spaghetti like strips of free neutron gas, before surround spherical "bubbles" of free neutron gas (Ravenhall et al, 1983). If we were to increase the density any more, we would reach the nuclear saturation density, at roughly $\rho_0 \approx 2.4 \times 10^{14} gcm^{-3}$, where nuclei "melt" into a sea of neutrons, protons, and electrons (Ravenhall et al, 1983). A simulation of these exotic shapes of matter can be found in Fig. 4.

5.2 Bethe-Johnson Equation of State

As we pass beyond the nuclear saturation density of $\rho_0 \approx 2.4 \times 10^{14} gcm^{-3}$, we enter into a region where nuclei have begun dissolving and merging together, known as the outer core of the neutron star. This high density regime is only understood to a density of roughly $2\rho_0$ (Acevedo et al, 2019), the density which corresponds to the end of the outer core, and the beginning of the inner core. In an effort to avoid lengthy N-body quantum mechanics calculations, we will investigate this region using the Bethe-Johnson equation of state. This equation of state utilizes the Yukawa potential, the theoretical potential of the nuclear force. Yukawa suggests that the meson particle mediates the exchange of the nuclear force, much like the photon mediates the exchange of the electromagnetic force (Griffiths, 2014). The potential is in the form

$$V = \pm g^2 \frac{e^{-\mu r}}{r} \tag{44}$$

where μ is the inverse Compton wavelength of the quanta, and g is the charge in some scalar or vector field.

Now, the Bethe-Johnson Equation of State. The exchange of mesons (mediating the nuclear force), creates the repulsive nucleon-nucleon interactions. The potential to first order for this equation of state is the Yukawa potential for a ω-meson. We take the range of this meson to be $\approx \mu_\omega^{-1} = \hbar/m_\omega c = 0.25 fm$ (Bethe & Johnson, 1974). Therefore, our potential is

$$V_\omega \equiv g_\omega^2 \frac{e^{-\mu r}}{r} \tag{45}$$

Where $g_\omega = (29.6\hbar c)^{1/2}$ (Bethe & Johnson, 1974). Avoiding tedious calculations, we can arrive at the pressure:

$$\frac{\epsilon}{n} \equiv 236 n^a + m_n c^2 \tag{46}$$

$$P = n^2 \frac{d(\epsilon/n)}{dn} \tag{47}$$

$$= 363.44 n^{a+1} MeV\ fm^{-3} \tag{48}$$

$$= 5.83 \times 10^{35} n^{a+1} dyne\ cm^{-2} \tag{49}$$

Where $a = 1.54$, and $0.1 \leq n \leq 3\ fm^{-3}$ which is equivallent to $1.7 \times 10^{14} \leq \rho \leq 1.1 \times 10^{16} gcm^{-3}$, with n being the number density of baryons found in Eqn. 36 (The Physics Of Compact Objects, 1983, Pearson et al, 2007, Bethe & Johnson, 1974).

This works fairly well to approximate the outer core, until we reach twice the nuclear saturation density. Below this point, physics changes once again.

6 Densities Below Twice Nuclear Saturation

Little is well understood below $2\rho_0 \approx 5 \times 10^{14} gcm^{-3}$. At this point, we can only provide well educated guesses. Many of the issues preventing researchers from making further progress include pion condensation, neutron and proton superfluidity, and N-body quantum mechanical simulations, to name just a few. Additionally, while we used the Yukawa potential to represent nuclear interactions in Sec. 5, the true form of the nuclear potential remains unknown. Some suggestions for the structure of the inner core are that it is comprised of quark matter, strange matter, or pion/kaon condensates (The Physics of Compact Objects). Luckily, the according to Collins and Perry (1975) quark matter can be treated to first order as a ideal relativistic Fermi gas (Collins & Perry, 1975).

In the paper "Superdense Matter: Neutrons or Asymptotically Free Quarks" by Collins and Perry, they mention that at high enough density, say a few ρ_0, there is a possibility that matter may undergo a phase transition in which quarks drip out of individual neutrons and protons, creating a free gas of quarks (Collins & Perry, 1975). We do not know the potential mediating quark interactions, so we treat the system as an ideal relativistic Fermi gas. Additionally, it must be noted that quarks may change "flavours" into other quarks by weak interactions (The Physics Of Compact Objects, 1983). We would only expect up, down, and strange quarks to undergo these changes, as they are the least massive. The Fermi momenta for these particle can be determined by requiring zero charge and the system to be in equilibrium for $d \leftrightarrow u + l + \bar{\nu}$ and $s \leftrightarrow u + l + \bar{\nu}$ where l is an electron or muon. We take the Fermi momentum of the neutrino to be $p_{F,\nu} = 0$ because it interacts so weakly, and require the Fermi momentum to be above the strange quark mass, therefore $p_F \gg M_s$. Under these conditions, we find

$$n_u = n_s = n_d, \quad n_e = n_\mu = 0 \tag{50}$$

Now, as a first order approximation, we examine the equations of state for a free-quark gas. Summing over the Fermi momentum of the up, down, and strange quarks, they are as follows,

$$P = (1/24)d\Sigma_i \frac{p_{F,i}^4}{\pi^2} \tag{51}$$

$$\rho = (1/8)d\Sigma_i \frac{p_{F,i}^4}{\pi^2} \tag{52}$$

Clearly, the pressure and density remain closely related. Without going into much detail, the value for the degeneracy factor, d, is 6 due to spin and colour degrees of freedom (Collins Perry, 1975). As $\rho \to \infty$, we find

$$P \ \to \ \frac{1}{3}\rho c^2 \tag{53}$$

We show the similar asymptotic equation of state for an ideal relativistic nucleon gas for contrast to be

$$P \ \to \ \rho c^2 \tag{54}$$

And so concludes our analysis in this section.

7 Appendix - A

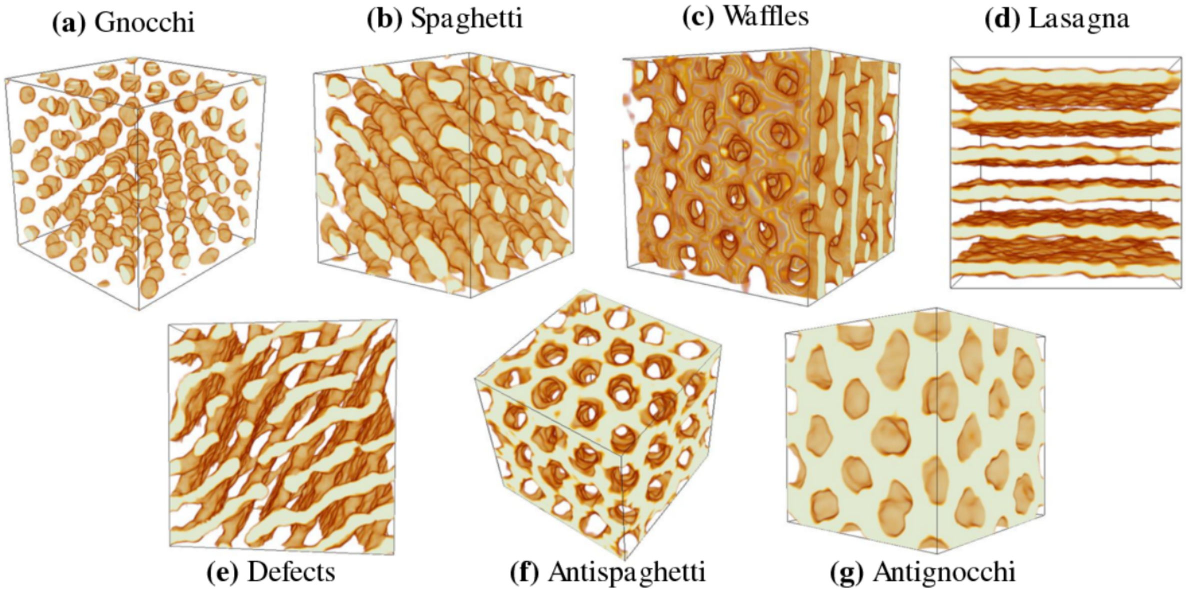

Figure 4: A simulation of nuclear pasta. From (Kycia et al, 2017).

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